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On Gibbs Measures of Models with Competing Ternary and Binary Interactions and Corresponding von Neumann Algebras

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In the present paper a model with competing ternary (J_2) and binary (J_1) interactions with spin values ± 1, on a Cayley tree is considered. One studies the structure of Gibbs measures for the model considered. It is known, that under some conditions on parameters J_1, J_2 (resp. in the opposite case) there are three (resp. a unique) translation-invariant Gibbs measures. We prove, that two of them (minimal and maximal) are extreme in the set of all Gibbs measures and also construct two periodic (with period 2) and uncountable number of distinct non-translation-invariant Gibbs measures. One shows that they are extreme. Besides, types of von Neumann algebras, generated by GNS-representation associated with diagonal states corresponding to extreme periodic Gibbs measures, are determined. Namely, it is shown that an algebra associated with the unordered phase is a factor of type III_λ, where $\lambda = \exp\{-2\beta J_2\}$, $\beta > 0$ is the inverse temperature. We find conditions, which ensure that von Neumann algebras, associated with the periodic Gibbs measures, are factors of type III_δ, otherwise they have type III_1.

KEY WORDS: Cayley tree; competing interactions; Gibbs measure; GNS-construction; Hamiltonian; von Neumann algebra.

1. INTRODUCTION

One of the central problems in the theory of Gibbs measures is the description of infinite-volume (or limiting) Gibbs measures corresponding to a given Hamiltonian. The existence of such measures for a wide class of

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Hamiltonians was established in the ground-breaking work of Dobrushin.[39] However, the complete analysis of the set of limiting Gibbs measures for a specific Hamiltonian is often a difficult problem. On a cubic lattice, for small values of $\beta = \frac{1}{T}$, where $T > 0$ is the temperature, a Gibbs measure is unique (refs. 33, 39) which reflects a physical fact that at high temperatures there is no phase transition. But the analysis for low temperatures requires specific assumptions about the model, in particular, about the form of the Hamiltonian. In particular, if there are only binary interactions then the problem of describing limiting Gibbs measures is more feasible. The classical example of such a model is the Ising model, with two values of spin ± 1. It was considered in refs. 33 and 44 and became actively researched in the 1990's and afterwards.[7–11, 36, 45] In the present paper we consider models with competing ternary and binary interactions on a Cayley tree. We note that the Ising models on a Cayley tree with competing interactions have been studied extensively (see refs. 25–27, 38) since the appearance of the Vannimenus model (see ref. 43), in which the physical motivations for the urgency of study such models was presented. In all of these works no exact solutions of the phase transition problem were found, but some solutions for specific parameter values were presented. In ref. 16 it has been proved that for the model with ternary and binary interactions a phase transition occurs for medium temperature values, which differs essentially from the well-known results for the ordinary Ising model, in which phase transition occurs at low temperature. It is known, that in the quantum statistical mechanics concrete systems are identified with states on corresponding algebras. In many cases the algebra can be chosen to be a quasi-local algebra of observables. The states on these algebras, satisfying KMS-condition, as is known, describe equilibrium states of the quantum system. On the other hand, for classical systems with the finite radius of interactions, limiting Gibbs measures are know to be Markov random fields. In connection with this, there arises a problem of constructing analogues of non-commutative Markov chains. In ref. 1 Accardi explored this problem, he introduced and studied non-commutative Markov states on the algebra of quasi-local observables, which were agreed with the classical Markov chains. In refs. 2 and 23 studied modular properties of the non-commutative Markov states.

The type analysis of the quasi-free factors (i.e., factors generated by quasi-free representations) has been an interesting problem since the appearance of the pioneering work of Araki and Wyss.[5] In ref. 31 was constructed a family of representations of uniformly hyperfinite algebras, which can be treated as a free quantum lattice system. In this case factors corresponding to these representations had type III_λ, $\lambda \in (0, 1)$. More general constructions of product states were considered in ref. 4.

In the case CAR-algebra, the rough classification into types I_∞, II_1, II_∞, and III was obtained in the 60's by several authors.[15, 32] In ref. 30 it has described the classification of type III quasi-free factors in terms of spectral properties of positive operators parameterizing quasi-free states. It is known,[3, 30] that every quasi-free state on CAR-algebra can be regarded as the product state on $A = \bigotimes_{n \geqslant 1} M_2(\mathbb{C})$.

Observe that the product states can be viewed as the Gibbs states of the Hamiltonian system in which interactions between particles of the system are absent, i.e., the system is a free lattice quantum spin system. So, it is interesting to consider the quantum lattice systems with non-trivial interactions, which leads us, as it was mentioned above, to the consideration of the Markov states. Simple examples of such systems are the Ising and Potts models, which have been studied in many papers.[36, 39] We note that all Gibbs states corresponding to these models are Markov random fields. Full type analysis of von Neumann algebras associated with the Markov states is still an open problem. In ref. 28 for the Ising model on a Cayley tree the types of factors corresponding to translation-invariant Gibbs states were found. In ref. 29 for a class of non-homogeneous Potts model it was proved that a von Neumann algebra associated with the unordered phase of this model is a factor of type III_1. Some particular cases of the Markov states were considered in ref. 22.

The present paper is devoted to the study the structure of the set of Gibbs measures of the model considered in ref. 16 and the type analysis of some class of Markov states, which correspond to these Hamiltonian systems. More precisely, we consider the extremity of Gibbs measures and determine the types of von Neumann algebras generated by GNS-representation associated with diagonal states corresponding to the extreme Gibbs measures. So, new examples of factors associated with physical systems will be constructed.

The paper is organized as follows.

In Section 2 we give some preliminary definitions of a model with competing ternary and binary interactions on a Cayley tree and corresponding Gibbs measures. Also, we recall some definitions from von Neumann algebras theory.

In Section 3 we reduce the problem of describing limit Gibbs measures to the problem of solving a nonlinear functional equation.

In Section 4 we consider translation-invariant Gibbs measures of the model. We prove that the minimum and maximum translation-invariant Gibbs measures are extreme. Two periodic (with period 2) and uncountable number of distinct non-translation-invariant Gibbs measures are constructed. It is shown that they are extreme.

In Section 5, we determine the types of von Neumann algebras generated by GNS-representation associated with diagonal states corresponding to the extreme periodic measures. Namely, it is shown, that an algebra associated with the unordered phase is a factor of type III_λ, where $\lambda = \exp\{-2\beta J_2\}$, $\beta > 0$ is the inverse temperature. We find conditions such that, if they are satisfied, then von Neumann algebras associated with the periodic Gibbs measures are factors of type III_δ, otherwise they have type III_1.

In the final Section 6 we discuss the obtained results.

2. DEFINITIONS AND PRELIMINARY RESULTS

The Cayley tree Γ^k (see ref. 6) of order $k \geqslant 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k, L is the set of edges of Γ^k and i is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called *nearest neighbouring vertices*, and we write $l = \langle x, y\rangle$. The distance $d(x, y)$, $x, y \in V$ on the Cayley tree is defined by the formula

$$d(x, y) = \min\{d \mid \exists x = x_0, x_1, \ldots, x_{d-1}, x_d = y \in V \text{ such that the pairs } \langle x_0, x_1\rangle, \ldots, \langle x_{d-1}, x_d\rangle \text{ are nearest neighbouring vertices}\}.$$

For the fixed $x^0 \in \Gamma^k$ we set

$$W_n = \{x \in V \mid d(x, x^0) = n\},$$

$$V_n = \bigcup_{m=1}^{n} W_m = \{x \in V \mid d(x, x^0) \leqslant n\},$$

$$L_n = \{l = \langle x, y\rangle \in L \mid x, y \in V_n\},$$

for an arbitrary point $x^0 \in V$. Denote $|x| = d(x, x_0)$, $x \in V$.

Three vertices $x, y, z \in V$ is called *ternary neighboring vertices* if $\langle x, y\rangle$ and $\langle y, z\rangle$ are nearest neighboring vertices, here $x, z \in W_n$ and $y \in W_{n-1}$ for some $n \in N$, and it is denoted by $\langle x, y, z\rangle$.

A collection of the pairs $\langle x, x_1\rangle, \ldots, \langle x_{d-1}, y\rangle$ is called a *path* from the point x to the point y. We write $x < y$ if the path from x^0 to y goes through x. Call vertex y a *direct successor* of x if $y > x$ and x, y are nearest neighbors. Denote by $S(x)$ the set of direct successors, i.e.,

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \qquad x \in W_n.$$

Observe that any vertex $x \neq x^0$ has k direct successors and x^0 has $k+1$.

Proposition 2.1 (see ref. 17). There exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geqslant 1$ and the group G_k of the free products of $k+1$ cyclic groups of the second order with generators $a_1, a_2,..., a_{k+1}$.

Let us define a group structure on the group G_k as follows. Vertices which corresponds to the "words" $g, h \in G_k$ are called nearest neighbors and are connected by an edge if either $g = ha_i$ or $h = ga_j$ for some i or j. The graph thus defined is a Cayley tree of order k.

Consider a left (resp. right) transformation shift on G_k defined as: for $g_0 \in G_k$ we put

$$T_{g_0} h = g_0 h \quad (\text{resp. } T_{g_0} h = h g_0) \qquad \forall h \in G_k.$$

It is easy to see that the set of all left (resp. right) shifts on G_k is isomorphic to the group G_k.

We consider models where the spin takes values in the set $\Phi = \{-1, 1\}$ and assigned to the vertices of the tree. A configuration σ on V is then defined as a function $x \in V \to \sigma(x) \in \Phi$; the set of all configurations coincides with $\Omega = \Phi^V$. The Hamiltonian is of competing ternary and binary model has the form

$$H(\sigma) = -J_2 \sum_{\langle x, y, z \rangle} \sigma(x)\,\sigma(y)\,\sigma(z) - J_1 \sum_{\langle x, y \rangle} \sigma(x)\,\sigma(y) \tag{2.1}$$

where $J_1, J_2 \in \mathbb{R}$ are coupling constants and $\sigma \in \Omega$.

We consider a standard σ-algebra $\mathscr{F}$ of subsets of Ω generated by cylinder subsets, all probability measures are considered on $(\Omega, \mathscr{F})$. A probability measure μ is called a *a Gibbs measure* (with Hamiltonian H) if it satisfies the DLR equation: $\forall n = 1, 2,...$ and $\sigma_n \in \Phi^{V_n}$:

$$\mu(\{\sigma \in \Omega : \sigma|_{V_n} = \sigma_n\}) = \int_\Omega \mu(d\omega)\, \nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n)$$

where $\nu^{V_n}_{\omega|_{W_{n+1}}}$ is the conditional probability

$$\nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n) = Z^{-1}(\omega|_{W_{n+1}}) \exp(-\beta H(\sigma_n \parallel \omega|_{W_{n+1}})).$$

where $\beta > 0$. Here $\sigma_n|_{V_n}$ and $\omega|_{W_{n+1}}$ denote the restriction of $\sigma, \omega \in \Omega$ to V_n and W_{n+1} respectively. Next, $\sigma_n : x \in V_n \to \sigma_n(x)$ is a configuration in V_n and $H(\sigma_n \parallel \omega|_{W_{n+1}})$ is defined as the sum $H(\sigma_n) + U(\sigma_n, \omega|_{W_{n+1}})$ where

$$H(\sigma_n) = -J_2 \sum_{\langle x, y, z\rangle:\, x, y, z \in V_n} \sigma_n(x)\, \sigma_n(y)\, \sigma_n(z)$$

$$-J_1 \sum_{\langle x, y\rangle:\, x, y \in V_n} \sigma_n(x)\, \sigma_n(y)$$

$$U(\sigma_n, \omega|_{W_{n+1}}) = -J_2 \sum_{\langle x, y, z\rangle:\, x, z \in W_{n+1},\, y \in W_n} \omega(x)\, \sigma_n(y)\, \omega(z)$$

$$-J_1 \sum_{\langle x, y\rangle:\, x \in V_n,\, y \in W_{n+1}} \sigma_n(x)\, \omega(y).$$

Finally, $Z(\omega|_{W_{n+1}})$ stands for the partition function in V_n with the boundary condition $\omega|_{W_{n+1}}$:

$$Z(\omega|_{W_{n+1}}) = \sum_{\tilde{\sigma}_n \in \Phi^{V_n}} \exp(-\beta H(\tilde{\sigma}_n \parallel \omega|_{W_{n+1}})).$$

It is known (see ref. 39) that for any sequence $\omega^{(n)} \in \Omega$, any limiting point of the measures $\tilde{\nu}^{V_n}_{\omega^{(n)}|_{W_{n+1}}}$ is a Gibbs measure. Here $\tilde{\nu}^{V_n}_{\omega^{(n)}|_{W_{n+1}}}$ is a measure on Ω such that $\forall n' > n$:

$$\tilde{\nu}^{V_n}_{\omega^{(n)}|_{W_{n+1}}}(\{\sigma \in \Omega: \sigma|_{V_{n'}} = \sigma_{n'}\}) = \begin{cases} \nu^{V_n}_{\omega^{(n)}|_{W_{n+1}}}(\sigma_{n'}|_{V_n}), & \text{if } \sigma_{n'}|_{V_{n'}\setminus V_n} = \omega^{(n)}|_{V_{n'}\setminus V_n} \\ 0, & \text{otherwise.} \end{cases}$$

We now recall some facts from von Neumann algebra theory. Let $B(H)$ be an algebra of all bounded linear operators on a Hilbert space H (over the field of complex numbers $\mathbb{C}$). A weak (operator) closed *-subalgebra $\mathcal{N}$ in $B(H)$ is called *von Neumann algebra* if it contains the identity operator ϵ. By $Proj(\mathcal{N})$ it is denoted the set of all projections in $\mathcal{N}$. A von Neumann algebra is a *factor* if its center $Z(\mathcal{N})$ $(= \{x \in \mathcal{N}: xy = yx, \forall y \in \mathcal{N}\})$ is trivial, i.e., $Z(\mathcal{N}) = \{\lambda \mathbb{1} \mid \lambda \in \mathbb{C}\}$. The von Neumann algebras split into the classes I (I_n, $n < \infty$, I_∞), II (II_1, II_∞), and III.[12]

An element $x \in \mathcal{N}$ is called positive if there is an element $y \in \mathcal{N}$ such that $x = y^* y$. A linear functional ω on $\mathcal{N}$ is called a *state* if $\omega(x^* x) \geqslant 0$ for all $x \in \mathcal{N}$ and $\omega(\mathbb{1}) = 1$. A state ω is said to be normal if $\omega(\sup_\alpha x_\alpha) = \sup_\alpha \omega(x_\alpha)$ for any bounded increasing net $\{x_\alpha\}$ of positive elements of $\mathcal{N}$. A state ω is called a *trace* (resp. *faithful*) if the condition $\omega(xy) = \omega(yx)$ holds for all $x, y \in \mathcal{N}$ (resp. if the equality $\omega(x^* x) = 0$ implies $x = 0$).

Let $\mathcal{N}$ be a factor, ω be a faithful normal state on $\mathcal{N}$ and σ^ω_t be the modular group associated with ω (see ref. 12, Definition 2.5.15). We let

$\Gamma(\sigma^\omega)$ denote the Connes spectrum of the modular group σ_t^ω (see ref. 14, Definition 2.2.1).

Definition (see ref. 14). The factor $\mathcal{N}$ is of type

(i) III_1, if $\Gamma(\sigma^\omega) = \mathbb{R}$;

(ii) III_λ, if $\Gamma(\sigma^\omega) = \{n \log \lambda, n \in \mathbb{Z}\}$, $\lambda \in (0, 1)$;

(iii) III_0, if $\Gamma(\sigma^\omega) = \{0\}$.

(See, e.g., refs. 12 and 42 for details of von Neumann algebras and the modular theory of operator algebras.)

3. CONSTRUCTION OF GIBBS MEASURES

In this section we give the construction of a special class of limiting Gibbs measures for the competing ternary and binary model on a Cayley tree.

Let $h: x \to \mathbb{R}$ be a real valued function of $x \in V$. Given $n = 1, 2,...$ consider the probability measure $\mu^{(n)}$ on Φ^{V_n} defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left\{-\beta H(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x)\right\}. \tag{3.1}$$

Here, as before, $\beta = \frac{1}{T}$ and $\sigma_n: x \in V_n \to \sigma_n(x)$ and Z_n is the corresponding partition function:

$$Z_n = \sum_{\tilde{\sigma}_n \in \Omega_{V_n}} \exp\left\{\beta H(\tilde{\sigma}_n) + \sum_{x \in W_n} h_x \tilde{\sigma}(x)\right\}.$$

The consistency condition for $\mu^{(n)}(\sigma_n)$, $n \geqslant 1$ is

$$\sum_{\sigma^{(n)}} \mu^{(n)}(\sigma_{n-1}, \sigma^{(n)}) = \mu^{(n-1)}(\sigma_{n-1}), \tag{3.2}$$

where $\sigma^{(n)} = \{\sigma(x), x \in W_n\}$.

Let $V_1 \subset V_2 \subset \cdots \bigcup_{n=1}^{\infty} V_n = V$ and $\mu_1, \mu_2,...$ be a sequence of probability measures on $\Phi^{V_1}, \Phi^{V_2},...$ satisfying the consistency condition (3.2). Then, according to the Kolmogorov theorem,[37] there is a unique limit Gibbs measure μ_h on Ω such that for every $n = 1, 2,...$ and $\sigma_n \in \Phi^{V_n}$ the following equality holds

$$\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu^{(n)}(\sigma_n). \tag{3.3}$$

The following statement describes conditions on h_x guaranteeing the consistency condition of measures $\mu^{(n)}(\sigma_n)$. In the sequel for the simplicity we consider the case $k=2$.

Theorem 3.1. The measures $\mu^{(n)}(\sigma_n)$, $n=1,2,\ldots$ satisfy the consistency condition (3.2) if and only if for any $x\in V$ the following equation holds:

$$h_x = \frac{1}{2}\log\frac{\theta_1^2\theta_2 e^{2(h_y+h_z)}+\theta_1(e^{2h_y}+e^{2h_z})+\theta_2}{e^{2(h_y+h_z)}+\theta_1\theta_2(e^{2h_y}+e^{2h_z})+\theta_1^2} \tag{3.4}$$

here $\theta_1=e^{2\beta J_1}$, $\theta_2=e^{2\beta J_2}$, and $\langle y,x,z\rangle$ are ternary neighbors.

Proof. *Necessity.* According to the consistency condition (3.2) we have

$$\begin{aligned}\frac{Z_{n-1}}{Z_n}\sum_{\sigma^{(n)}}\exp\Bigg\{&-\beta H_{n-1}(\sigma_{n-1})+\beta J_1\sum_{x\in W_{n-1}}\sigma(x)(\sigma(y)+\sigma(z))\\ &+\beta J_2\sum_{x\in W_{n-1}}\sigma(x)\,\sigma(y)\,\sigma(z)+\sum_{x\in W_{n-1}}\sum_{y\in S(x)}h_y\sigma(y)\Bigg\}\\ =\exp\Bigg\{&-\beta H_{n-1}(\sigma_{n-1})+\sum_{x\in W_{n-1}}h_x\sigma(x)\Bigg\}.\end{aligned} \tag{3.5}$$

Whence we get

$$\begin{aligned}&\frac{Z_{n-1}}{Z_n}\sum_{\sigma^{(n)}}\prod_{x\in W_{n-1}}\exp\{\sigma(x)[\beta J_1(\sigma(y)+\sigma(z))+\beta J_2\sigma(y)\,\sigma(z)]+h_y\sigma(y)+h_z\sigma(z)\}\\ &=\prod_{x\in W_{n-1}}\exp\{h_x\sigma(x)\}.\end{aligned} \tag{3.6}$$

Let $\sigma_x^{(n)}=\{\sigma(y),\sigma(z)\}$, $x\in W_{n-1}$. Then it is easy to see that $\sigma^{(n)}=\bigcup_{x\in W_{n-1}}\sigma_x^{(n)}$. Hence

$$\begin{aligned}&\frac{Z_{n-1}}{Z_n}\prod_{x\in W_{n-1}}\sum_{\sigma_x^{(n)}}\exp\{\sigma(x)[\beta J_1(\sigma(y)+\sigma(z))+\beta J_2\sigma(y)\,\sigma(z)]+h_y\sigma(y)+h_z\sigma(z)\}\\ &=\prod_{x\in W_{n-1}}\exp\{h_x\sigma(x)\}.\end{aligned} \tag{3.7}$$

Now fix $x \in W_{n-1}$ and rewrite (3.7) for the cases $\sigma(x)=1$ and $\sigma(x)=-1$ then we can find

$$\frac{\sum_{\sigma_x^{(n)}=\{\sigma(y),\sigma(z)\}} \exp\{\beta J_1(\sigma(y)+\sigma(z))+\beta J_2\sigma(y)\,\sigma(z)+h_y\sigma(y)+h_z\sigma(z)\}}{\sum_{\sigma_x^{(n)}=\{\sigma(y),\sigma(z)\}} \exp\{-\beta J_1(\sigma(y)+\sigma(z))-\beta J_2\sigma(y)\,\sigma(z)+h_y\sigma(y)+h_z\sigma(z)\}} = \exp\{2h_x\}. \tag{3.8}$$

Denote

$$\begin{aligned} W_1 &= \exp(2J_1\beta+J_2\beta+h_y+h_z)+\exp(-J_2\beta-h_y+h_z) \\ &\quad +\exp(-J_2\beta+h_y-h_z)+\exp(-2J_1\beta+J_2\beta-h_y-h_z) \\ W_{-1} &= \exp(-2J_1\beta-J_2\beta+h_y+h_z)+\exp(J_2\beta-h_y+h_z) \\ &\quad +\exp(J_2\beta+h_y-h_z)+\exp(2J_1\beta-J_2\beta-h_y-h_z). \end{aligned}$$

It then follows from (3.8) that

$$\exp\{2h_x\} = \frac{W_1}{W_{-1}}. \tag{3.9}$$

The equality (3.9) implies (3.4).

Sufficiency. From (3.6), (3.7), (3.8), (3.9), we get (3.4) and hence (3.2). The theorem is proved.

According to Theorem 3.1 the problem of describing of Gibbs measures is reduced to the description of solutions of the functional equation (3.4).

4. EXTREMITY OF GIBBS MEASURES

4.1. Extremity of Translation-Invariant Gibbs Measures

This subsection is devoted to translation-invariant Gibbs measures and we consider a problem of extremity ones.

According to Proposition 2.1 any transformation S of the group G_k induces a shift automorphism $\tilde{S}: \Omega \to \Omega$ by

$$(\tilde{S}\sigma)(h) = \sigma(Sh), \qquad h \in G_k, \quad \sigma \in \Omega.$$

By $\mathscr{G}_k$ we denote the set of all shifts of Ω.

We say that a Gibbs measure μ on Ω is *translation-invariant* if for any $T \in \mathscr{G}_k$ the equality $\mu(T(A)) = \mu(A)$ is valid for all $A \in \mathscr{F}$.

The analysis of the solutions of (3.4) is rather tricky. It is natural to begin with translation-invariant solutions where $h_x = h$ is constant for all $x \in V$. It is clear that a Gibbs measure corresponding to this solution is translation-invariant. This case has been investigated in ref. 16.

In this case from (3.4) we virtue

$$u = \frac{\theta_1^2\theta_2 u^2 + 2\theta_1 u + \theta_2}{u^2 + 2\theta_1\theta_2 u + \theta_1^2} \tag{4.1}$$

where $u = e^{2h}$.

Denote $\eta = \theta_1(2-\theta_1)$, and

$$Q = -\frac{4\eta^3\theta_2^4 + (\eta^4 + 18\eta^2 - 27)\theta_2^2 + 4\eta^3}{108}.$$

Proposition 4.1. (see ref. 16). At $\theta_1 > 3$ for all pairs (θ_1, θ_2) such that $Q < 0$ Eq. (4.1) has three positive solutions $u_1^* < u_2^* < u_3^*$. Otherwise Eq. (4.1) has a unique solution u_*.

By μ_1, μ_2, μ_3 we denote Gibbs measures corresponding to these solutions. As a consequence of Proposition 4.1 we can formulate the following

Corollary 4.2. At $\theta_1 > 3$ and $Q < 0$ there are three translation-invariant Gibbs measures μ_1, μ_2, μ_3.

Denote $u_x = \exp(2h_x)$, $x \in V$. Then the functional equation (3.4) is rewritten as follows

$$u_x = \frac{\theta_1^2\theta_2 u_y u_z + \theta_1(u_y + u_z) + \theta_2}{u_y u_z + \theta_1\theta_2(u_y + u_z) + \theta_1^2} \tag{4.2}$$

here $\langle y, x, z\rangle$ are ternary neighboring vertices.

Proposition 4.3. Let $\theta_1 > 3$, $Q < 0$ and u_x be a solution of Eq. (4.2). Then

$$u_1^* \leqslant u_x \leqslant u_3^* \qquad \text{for any} \quad x \in V.$$

Proof. It is clear that $u_x > 0$, $\forall x \in V$. Put

$$f(x, y) = \frac{\theta_1^2\theta_2 xy + \theta_1(x+y) + \theta_2}{xy + \theta_1\theta_2(x+y) + \theta_1^2}, \qquad x, y > 0. \tag{4.3}$$

Observe that the system

$$\begin{cases} \dfrac{\partial f(x,y)}{\partial x} = 0 \\ \dfrac{\partial f(x,y)}{\partial y} = 0 \end{cases}$$

has solutions only on the line $x = y$. Therefore, consider a function $g(x) = f(x,x)$. One may show that the function $g(x)$ is increasing on $(0,\infty)$ at $\theta_1 > 3$ and $Q < 0$. Hence we conclude that $\theta_2/\theta_1^2 < f(x,y) < \theta_1^2\theta_2$, for all $x, y > 0$. Now we consider the function on $x, y \in (\theta_2/\theta_1^2, \theta_1^2\theta_2)$. By similar reason as above we get

$$g\left(\frac{\theta_2}{\theta_1^2}\right) < f(x,y) < g(\theta_1^2\theta_2).$$

Repeating this argument one gets

$$g^{(n)}\left(\frac{\theta_2}{\theta_1^2}\right) < f(x,y) < g^{(n)}(\theta_1^2\theta_2),$$

for all $n \geqslant 1$. Here $g^{(n)}$ is the n-th iterate of the map $x \to g(x)$. The sequence $g^{(n)}(\theta_1^2\theta_2)$ is decreasing and bounded below by u_3^*. Its limit is a fixed point of g and thus equal to u_3^*. This proves that $u_x < u_3^*$. The lower estimate of u_x is found in a similar manner. This completes the proof.

Theorem 4.4. For the model (2.1) with parameters $J_1, J_2 \in \mathbb{R}$ on the Cayley tree Γ^2 the following assertions hold true

(i) if $\theta_1 > 3$, $Q < 3$ then the measures μ_1 and μ_3 are extreme;

(ii) in the opposite case there is a unique Gibbs measure $\mu_*(=\mu_2)$, i.e., there is no phase transition.

Proof. (i) Using Proposition 4.3 and by similar argument as in the proof of Theorem 12.31[21] we can show the extremity of measures μ_1, μ_3.

(ii) In this case Proposition 4.1 and 4.3 imply that $u_1^* = u_2^* = u_3^* = u_*$. Hence we have only Gibbs measure and according to Theorem 12.6[21] we conclude that the measure μ_* is extreme. The theorem is proved.

Remark. We note that at $J_1 = 0$ according to Theorem 4.4 for the considered model there is a unique Gibbs measure μ_0. This measure corresponds to the solution $h_x = 0$, $x \in V$, moreover it is extreme and the unordered phase, i.e., the spin $\sigma(x)$ takes its values ± 1 with respect to μ_0 with probability $1/2$.

4.2. Extremity of Periodic Gibbs Measures

Let G_k be a free product of $k+1$ cyclic groups of order two. According to Proposition 2.1 there is a one-to-one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k. Let $\hat{G}_k \subset G_k$ be a normal subgroup of finite index.

Definition. We say that $h = \{h_x \colon x \in G_k\}$ is $\hat{G}_k$-periodic if $h_{yx} = h_x$ for all $x \in G_k$ and $y \in \hat{G}_k$.

A Gibbs measure is called $\hat{G}_k$*-periodic* if it corresponds to $\hat{G}_k$-periodic function h.

Observe that a translation-invariant Gibbs measure is G_k-periodic.

In the sequel we consider the group G_2.

Denote

$$G_2^{(2)} = \{x \in G_2 \colon \text{the length of word } x \text{ is even}\}.$$

In this subsection we will construct $G_2^{(2)}$-periodic Gibbs measures and show that they are extreme.

A $G_2^{(2)}$-periodic Gibbs measure corresponds to function h_x defined by

$$h_x = \begin{cases} h_1, & \text{if} \quad x \in G_2^{(2)}, \\ h_2, & \text{if} \quad x \in G_2 \setminus G_2^{(2)}. \end{cases} \tag{4.4}$$

According to Theorem 3.4 function defined by this fashion must satisfy Eq. (3.4) in our case that equation has a form:

$$\begin{cases} u = \dfrac{\theta_1^2\theta_2 v^2 + 2\theta_1 v + \theta_2}{v^2 + 2\theta_1\theta_2 v + \theta_1^2}, \\[2ex] v = \dfrac{\theta_1^2\theta_2 u^2 + 2\theta_1 u + \theta_2}{u^2 + 2\theta_1\theta_2 u + \theta_1^2}, \end{cases} \tag{4.5}$$

where $u = \exp\{2h_1\}$, $v = \exp\{2h_2\}$.

The analysis of Eq. (4.5) is carried in the following

Proposition 4.5. If $\theta_1 \in (0; \sqrt{5}-2)$ and $\theta_2 \in (0; \sqrt{t_1}) \cup (\sqrt{t_2}; +\infty)$ or $\theta_1 \in (\sqrt{5}-2; \sqrt{2}-1)$ and $\theta_2 > 0$ then the equation (4.5) has two solutions (u_*, v_*), (v_*, u_*). Here

$$t_{1,2} = \frac{(\theta_1+1)^3(\theta_1^2+2\theta_1-1)^2(\theta_1+1\pm\sqrt{(\theta_1^2-1)(\theta_1^2+4\theta_1+1)(\theta_1^2+4\theta_1-1)})}{8\theta_1^3(\theta_1+2)}$$
$$-\frac{4\theta_1^2(\theta_1^2(\theta_1+1)^2+1)}{8\theta_1^3(\theta_1+2)}$$

and u_*, v_* are the solutions of the equation:

$$\theta_1^2((\theta_1\theta_2)^2+2\theta_1\theta_2^2+1)\,x^2+\theta_2(4\theta_1^3+\theta_1^4+4\theta_1^2-1)\,x+\theta_1^2(\theta_1^2+2\theta_1+\theta_2^2)=0. \tag{4.6}$$

Proof. If $u=v$ then it is clear that we get a translation-invariant Gibbs measure. To obtain the periodic measures we assume $u\neq v$. From (4.5) we find

$$\begin{aligned}\theta_1^2((\theta_1\theta_2)^2+2\theta_1\theta_2^2+1)\,u^5-\theta_2((\theta_1^3\theta_2)^2-4(\theta_1^2\theta_2)^2-6\theta_1^3-4\theta_1^2+1)\,u^4\\ -2\theta_1((\theta_1^2\theta_2)^2-\theta_1^3-4(\theta_1\theta_2)^2-\theta_1\theta_2^2+\theta_2^2)\,u^3\\ +2\theta_1\theta_2(\theta_1^4-\theta_1^3\theta_2^2-4\theta_1^2-\theta_1+1)\,u^2\\ +(\theta_1^6-4\theta_1^4-6\theta_1^3\theta_2^2-4(\theta_1\theta_2)^2+\theta_2^2)\,u-\theta_1^2\theta_2(\theta_1^2+2\theta_1+\theta_2^2)=0.\end{aligned}$$

From the last equation we get

$$\begin{aligned}&[u^3-\theta_1\theta_2(\theta_1-2)\,u^2+\theta_1(\theta_1-2)\,u-\theta_2]\\ &\quad\times[\theta_1^2((\theta_1\theta_2)^2+2\theta_1\theta_2^2+1)\,u^2+\theta_2(4\theta_1^3+\theta_1^4+4\theta_1^2-1)\,u+\theta_1^2(\theta_1^2+2\theta_1+\theta_2^2)]\\ &=0.\end{aligned}$$

Observe that the solutions of equation

$$u^3-\theta_1\theta_2(\theta_1-2)\,u^2+\theta_1(\theta_1-2)\,u-\theta_2=0 \tag{4.7}$$

describe only translation-invariant measures (see (4.1)). Hence the periodic measures correspond to the solutions of Eq. (4.6).

Full analysis of Eq. (4.6) shows that parameters θ_1, θ_2 must satisfy the condition of the proposition. This completes the proof.

By μ_{12} and μ_{21} we denote $G_2^{(2)}$-periodic measures corresponding to the solutions (u_*, v_*) and (v_*, u_*) respectively.

Thus we have the following

Theorem 4.6. For the model (2.1) the following assertions hold:

(i) If $J_1 > 0$ then $G_2^{(2)}$-periodic Gibbs measures coincide with translation - invariant Gibbs measures.

(ii) If $J_1 < 0$ and all conditions of Proposition 4.6 be satisfied then there are three $G_2^{(2)}$-periodic Gibbs measures μ_{12}, μ_{21}, and μ_*. Here μ_* is the measure corresponding to the unique solution of Eq. (4.7).

The proof immediately follows from Propositions 4.1 and 4.5.

Remark. We note that measure μ_* is a translation-invariant and measures μ_{12}, μ_{21} are not.

The next theorem describes the extremity of μ_{12}, μ_{21}.

Theorem 4.7. Let all conditions of Proposition 4.5 be satisfied. Then the measures μ_{12}, μ_{21} are extreme.

Proof. We note that in this case Proposition 4.3 is also valid. By argument similar to the proof of Theorem 4.4 one can show that the measures μ_{12}, μ_{21} are extreme.

4.3. Non Translation-Invariant Gibbs Measures

In this section we consider the case $\theta_1 > 3$, $\theta_2 > 0$. We use the measures μ_1, μ_3 to show that Eq. (3.4) admits uncountably many non-translation-invariant solutions.

Take an arbitrary infinite path $\pi = \{x_0, x_1, \ldots\}$ on the Cayley tree starting at the origin x^0: $x_0 = x^0$. We will establish a 1-1 correspondence between such paths and real numbers $t \in [0; 1]$.[8, 19, 35] We will map the path π to a function $h^\pi : x \in V \to h_x^\pi$ satisfying (3.4). Path π splits Cayley tree Γ^2 into two components Γ_1^2 and Γ_2^2.

Function h^π is then defined by

$$h_x^\pi = \begin{cases} \log u_1^*, & \text{if} \quad x \in \Gamma_1^2 \\ \log u_3^*, & \text{if} \quad x \in \Gamma_2^2. \end{cases} \tag{4.8}$$

Denote $F(x, y) = \frac{1}{2} \log f(\exp(2x), \exp(2y))$, here f is defined by (4.3).

Proposition 4.8. If $\theta_2 > 1$ then there is a number $0 < \gamma = \gamma(\theta_1, \theta_2) < 1$ such that the following inequality holds

$$|F(x_1, y) - F(x_2, y)| \leqslant \gamma\, |x_1 - x_2| \qquad \text{for any} \quad x_1, x_2, y \in \mathbb{R}.$$

Proof. Straightforward.

With the help of Proposition 4.8 it is easy to prove the following Theorem 4.9 similar to Theorem 3 of ref. 35.

Theorem 4.9. For any infinite path π there exists a unique function h^{π} satisfying (3.4), (4.8).

In the standard way (see refs. 8, 19, 34, 35) one can prove that functions $h^{\pi(t)}$ are different for different $t \in [0; 1]$.

Now let $\mu(t)$ denote the Gibbs measure corresponding to the function $h^{\pi(t)}$, $t \in [0; 1]$.

Using Theorem 4.4, similar to analogous theorem of ref. 8 we can prove the following

Theorem 4.10. For any $t \in [0; 1]$, there exists a unique extreme Gibbs measure $\mu(t)$. Moreover, the above Gibbs measures μ_1, μ_3 are specified as $\mu(0) = \mu_3$ and $\mu(1) = \mu_1$.

Because the measures $\mu(t)$ are different for different $t \in [0; 1]$ we obtain a continuum of distinct extreme Gibbs measures.

Remark. If we consider the case (ii) of Theorem 4.6 an analogous result as Theorem 4.10 one can prove with the aid of the periodic measures μ_{12} and μ_{21}.

5. DIAGONAL STATES GENERATED BY GIBBS MEASURES AND CORRESPONDING VON NEUMANN ALGEBRAS

5.1. Diagonal States Associated with the Unordered Phase μ_0.

In this subsection we consider a case $J_1 = 0$ and we determine a type of von Neumann algebra generated by the GNS-representation associated with the diagonal state corresponding to the unordered phase μ_0.

Consider C^*-algebra $A = \bigotimes_{\Gamma^k} M_2(\mathbb{C})$, where $M_2(\mathbb{C})$ is the algebra of 2×2 matrices over the field $\mathbb{C}$ of complex numbers. By e_{ij}, $i, j \in \{1, 2\}$ one denotes the basis matrices of the algebra $M_2(\mathbb{C})$. We let $CM_2(\mathbb{C})$ denote the commutative subalgebra of $M_2(\mathbb{C})$ generated by the elements e_{ii} $i \in \{1, 2\}$. We set $CA = \bigotimes_{\Gamma^k} CM_2(\mathbb{C})$. Elements of commutative algebra CA may be regarded as functions on the space $\Omega = \{e_{11}, e_{22}\}^{\Gamma^k}$. Given a measure μ on the measurable space (Ω, B), where B is the σ-algebra generated by cylindrical subsets of Ω. We construct a state ω_μ on A as follows. We set $\omega_\mu(x) = 0$ if the tensor monomial x of the basis matrices e_{ij}, $i, j \in \{1, 2\}$ contains at least one partial isometry. If $x \in CA$, we set $\omega_\mu(x) = \int_\Omega x \, d\mu$. The state thus obtained was introduced in ref. 42 and was said to be *diagonal*. In other words, if $P\colon A \to CA$ is a conditional expectation, then the state ω_μ can be defined by $\omega_\mu(x) = \mu(P(x))$, $x \in A$, here $\mu(P(x))$ means the integral of a function $P(x)$ under measure μ, i.e., $\mu(P(x)) = \int_\Omega P(x)(s)\, d\mu(s)$.

By ω_0 we denote the diagonal state generated by the unordered phase μ_0. On the finite dimensional C^*-subalgebra $A_{V_n} = \bigotimes_{V_n} M_2(\mathbb{C}) \subset A$ we rewrite the state ω_0 as follows

$$\omega_0(x) = \frac{tr(e^{\tilde{H}(V_n)} x)}{tr(e^{\tilde{H}(V_n)})}, \qquad x \in A_{V_n}, \tag{5.1}$$

where tr is a trace on A_{V_n}. The term $\sigma(x)\,\sigma(y)\,\sigma(z)$ in (3.2) we represent as a diagonal element of $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ in the standard basis as follows

$$\sigma(x)\,\sigma(y)\,\sigma(z) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{5.2}$$

Using (5.2), the form of Hamiltonian (2.1), (3.1), and (5.1) the Hamiltonian $\tilde{H}(V_n)$ in the standard basis of A_{V_n} has the form

$$\tilde{H}(V_n) = \sum_{\langle x, y, z\rangle :\, x, y, z \in V_n} F_{\langle x, y, z\rangle},$$

here and below

$$F_{\langle x, y, z\rangle} = \begin{pmatrix} A & O & O & O \\ O & B & O & O \\ O & O & B & O \\ O & O & O & A \end{pmatrix}, \quad A = \begin{pmatrix} \log p_1 & 0 \\ 0 & \log p_2 \end{pmatrix}, \quad B = \begin{pmatrix} \log p_2 & 0 \\ 0 & \log p_1 \end{pmatrix}, \tag{5.3}$$

$$p_1 = \frac{1}{e^{-2\alpha} + 1}, \qquad p_2 = \frac{e^{-2\alpha}}{e^{-2\alpha} + 1}, \qquad \alpha = \beta J_2. \tag{5.4}$$

Hence the state ω_0 is an extreme Gibbs state for quantized Hamiltonian

$$\tilde{H} = \sum_{\langle x, y, z\rangle} F_{\langle x, y, z\rangle}.$$

Denote $\mathcal{M} = \pi_{\omega_0}(A)''$, where π_{ω_0} is a GNS-representation associated with ω_0 (see ref. 12, Definition 2.3.18). Our goal in the present section is to determine a type of $\mathcal{M}$.

Remark. In ref. 42 general properties of a representation associated with diagonal state were studied, but concrete constructions of states were not considered there. In ref. 30 a deep classification of types of the factors generated by quasi-free states has been obtained. For translation-invariant Markov states the corresponding type analysis has been made in ref. 18.

According to the extremity of ω_0 (see ref. 13, Theorem 5.3.30) we find that $\mathcal{M}$ is a factor.

We note that the modular group of $\mathcal{M}$ associated with ω_0 is defined by

$$\sigma_t^{\omega_0}(x) = \lim_{V_n \to V} \exp\{it\tilde{H}(V_n)\}\, x \exp\{-it\tilde{H}(V_n)\}, \qquad x \in \mathcal{M} \tag{5.5}$$

here as before $\tilde{H}(\Lambda) = \sum_{\langle x, y, z\rangle :\, x, y, z \in V_n} F_{\langle x, y, z\rangle}$. For the last limit to exist we must show that a norm of potential $\tilde{H}$ is finite (see ref. 13, Theorem 6.2.4). First of all we recall the definition of a norm of a potential $\Psi = \sum_{X \subset \Gamma^k} \Psi(X)$ as follows:

$$\|\Psi\|_d = \sum_{n \geqslant 0} e^{dn} \left(\sup_{x \in \Gamma^k} \sum_{x \in X,\, |X| = n+1} \|\Psi(X)\| \right),$$

where $d > 0$. Here $\Psi(X) \in A_X = \bigotimes_X M_2(\mathbb{C})$.

Now we compute $\|\tilde{H}\|_d$:

$$\begin{aligned}
\|\tilde{H}\|_d &= e^{2d} \left(\sup_{x \in \Gamma^k} \sum_{x \in X,\, X = \{u, v, w\}} \|F_{\langle u, v, w\rangle}\| \right) \\
&= k e^{2d} \sup_{\{u, v, w\} \in L} \|F_{\langle u, v, w\rangle}\| = k e^{2d} \max_i |\log p_i| < \infty.
\end{aligned}$$

Hence the norm of $\tilde{H}$ is finite, therefore the limit in (5.5) exists.

By $\mathcal{M}^\sigma$ one denotes the centralizer of ω_0, which is defined as

$$\mathcal{M}^\sigma = \{x \in \mathcal{M} : \sigma_t^{\omega_0}(x) = x, \text{ for all } t \in \mathbb{R}\}.$$

Lemma 5.1. For the modular group $\sigma_t^{\omega_0}$ and the number $t_0 = -2\pi/\log\lambda$, where $\lambda = \exp\{-2\alpha\}$ the equality holds

$$\sigma_{t_0}^{\omega_0} = Id,$$

here and below Id is the identity mapping and $\alpha = \beta J_2$.

Proof. From (5.3) and (5.4) we have

$$\exp(itF_{\langle x,y,x\rangle}) = \begin{pmatrix} \exp(itA) & O & O & O \\ O & \exp(itB) & O & O \\ O & O & \exp(itB) & O \\ O & O & O & \exp(itA) \end{pmatrix}$$

$$= \frac{1}{(\lambda+1)^{it}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^{it} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^{it} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^{it} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^{it} \end{pmatrix}.$$

Then from (5.5) and the last equality we find that $\sigma_{t_0}^{\omega_0} = Id$. This completes the proof.

Let us prove the following useful

Proposition 5.2. Let $\mathcal{N}$ be a factor, φ be a faithful normal state on $\mathcal{N}$ and let σ_t^{φ} be its modular group. If for the number $t_0 = -2\pi/\log\lambda$ the equality $\sigma_{t_0}^{\varphi} = Id$ is valid then $\mathcal{N}$ can not be a factor of type III_1.

Proof. Let us assume that $\mathcal{N}$ be a factor of type III_1. Denote $\alpha = \sigma_{t_0}^{\varphi}$. According to the Lemma 2.9[24] the crossed-product $\mathcal{N} \times_\alpha \mathbb{Z}$ is a factor of type III_λ.

On the other hand, according to Section 22.6 in ref. 41 every element

$$x_A = \sum_n \pi_\alpha(a(n))\, u_n, \qquad A = \{a(n)\} \subset \mathcal{N},$$

of the crossed-product $\mathcal{N} \times_\alpha \mathbb{Z}$ belongs to the center of this algebra if and only if the following relations hold:

(i) $a(n)\,\alpha^n(x) = xa(n)$ for all $x \in \mathcal{N}$, $n \in \mathbb{Z}$;

(ii) $\alpha^n(a(m)) = a(m)$ for all $m, n \in \mathbb{Z}$.

Here π_α and u_n are related to the crossed-product.

Now define a sequence $A_D = \{a_D(n)\}$ as follows

$$a_D(n) = \begin{cases} \mathbb{1}, & \text{if } \; |n| \leqslant D \\ 0, & \text{if } \; |n| > D. \end{cases}$$

Here D is any fixed positive integer.

Then it is easy to see that the conditions (i) and (ii) for the element x_{A_D} is fulfilled. So the element x_{A_D} belongs to the center. But $x_{A_D} \notin \mathbb{C}\mathbb{1}$. This contradicts to the factority of $\mathcal{N} \times_\alpha \mathbb{Z}$. The proposition is thus proved.

Since the state ω_0 is translation-invariant then, due to Corollary 4.3 of ref. 40, the factor $\mathcal{M}$ has type either III_δ or III_1. According to Lemma 5.1 and Proposition 5.2 we conclude that factor $\mathcal{M}$ has type III_δ for some $\delta \in (0, 1)$. This means that the Connes' spectrum $\Gamma(\sigma^{\omega_0})$ of σ^{ω_0} is the set $\{n \log \delta \colon n \in \mathbb{Z}\}$, i.e., $\Gamma(\sigma^{\omega_0})$ is discrete. Then using Proposition 16.4 of ref. 41, we find that the centralizer $\mathcal{M}^\sigma$ is a factor. Now we are going to show that $\delta = \lambda$. To do this we compute the Connes' spectrum.

It is known (see ref. 14, Proposition 2.2.2) that Connes' spectrum $\Gamma(\alpha)$ of a group of automorphisms $\alpha = \{\alpha_g\}_{g \in G}$ of von Neumann algebra $\mathcal{N}$ has the following form

$$\Gamma(\alpha) = \cap \{Sp(\alpha^e) \mid e \in Proj(Z(\mathcal{N}^\alpha)),\; e \neq 0\}, \tag{5.6}$$

where $\alpha^e(x) = \alpha(exe)$, $x \in e\mathcal{N}e$ and $Z(M^\alpha)$ is the center of subalgebra

$$\mathcal{N}^\alpha = \{x \in \mathcal{N} \colon \alpha_g(x) = x, \forall g \in G\}.$$

Here $Sp(\alpha)$ be the Arveson's spectrum of a group of automorphisms α (for more details, see refs. 14, 41).

Above we have just proved that $\mathcal{M}^\sigma$ is a factor, this means $Z(\mathcal{M}^\sigma) = \mathbb{C}\mathbb{1}$. Then the equality (5.6) implies $\Gamma(\sigma^{\omega_0}) = Sp(\sigma^{\omega_0})$.

We now consider the operator $\tilde{H}(V_n) = \sum_{\langle x, y, z \rangle \colon x, y, z \in V_n} F_{\langle x, y, z \rangle}$. We let $Sp(\tilde{H}(V_n))$ denote the spectrum of the operator $\tilde{H}(V_n)$. Setting

$$\sigma_t^{\omega_0, n}(x) = \exp\{it\tilde{H}(V_n)\}\, x \exp\{-it\tilde{H}(V_n)\}, \qquad x \in \mathcal{M},$$

we obtain

$$Sp(\sigma^{\omega_0, n}) = Sp(\tilde{H}(V_n)) - Sp(\tilde{H}(V_n)) = \{p - q \colon p, q \in Sp(\tilde{H}(V_n))\}. \tag{5.7}$$

From (5.3) it is evident that $\log p_i \in Sp(H_{V_n})$, $i = 1, 2$. Then (5.7) implies that $Sp(\sigma^{\omega_0, n})$ is generated by the elements

$$\log\left(\frac{p_i}{p_j}\right), \qquad i, j = 1, 2.$$

From (5.4) we have

$$\frac{p_i}{p_j} = \begin{cases} 1, & \text{if} \quad i = j, \\ \lambda^{-1}, & \text{if} \quad i > j, \\ \lambda, & \text{if} \quad j > i, \end{cases}$$

here as before $\lambda = e^{-2\alpha}$. From this we get

$$Sp(\sigma^{\omega_0, n}) = \{n \log \lambda\}_{n=-m}^{m}.$$

Hence we obtain

$$\Gamma(\sigma^{\omega_0}) = \{n \log \lambda\}_{n \in \mathbb{Z}}.$$

Consequently, we find that $\mathcal{M}$ is a factor of type III_λ.

So we have just proved the following

Theorem 5.3. A von Neumann algebra $\mathcal{M}$ corresponding to the unordered phase of the model with ternary and binary interactions (2.1) with $J_1 = 0$ on the Cayley tree Γ^2 is a factor of type III_λ, where $\lambda = \exp\{-2\beta J_2\}$.

5.2. Diagonal States Associated with Periodic Extreme Gibbs Measures

In this subsection we deal with the extreme $G_2^{(2)}$-periodic Gibbs measures constructed in Section 4 ($J_1, J_2 \neq 0$).

Let μ be an extreme $G_2^{(2)}$-periodic Gibbs measure, i.e., μ is either translation-invariant or $G_2^{(2)}$-periodic. By ω_μ we denote the associated diagonal state on C^*-algebra A. The diagonal state ω_μ as in previous subsection is rewritten on A_{V_n} as follows

$$\omega_\mu(x) = \frac{tr(e^{\tilde{H}_\mu(V_n)} x)}{tr(e^{\tilde{H}_\mu(V_n)})}, \qquad x \in A_{V_n}. \tag{5.8}$$

By the reasoning similar to the one in the previous subsection we can write the form of Hamiltonian $\tilde{H}_\mu(V_n)$ as follows:

$$\tilde{H}_\mu(V_n) = \sum_{\langle x,y,z\rangle: x,y,z\in V_n} F_{\langle x,y,z\rangle} + \sum_{\langle x,y\rangle: x,y\in V_n} \Phi_{\langle x,y\rangle} + \sum_{x\in W_n} h_x \sigma_x^z,$$

here as before $F_{\langle x,y,z\rangle}$ is defined in (5.3), h_x is a solution of Eq. (3.4) (see Section 4) and

$$\Phi_{\langle x,y\rangle} = \begin{pmatrix} A_1 & O \\ O & B_1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} \log p_{11} & 0 \\ 0 & \log p_{22} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \log p_{22} & 0 \\ 0 & \log p_{11} \end{pmatrix}, \tag{5.9}$$

$$p_{11} = \frac{1}{\lambda_1+1}, \quad p_{22} = \frac{\lambda_1}{\lambda_1+1}, \quad \lambda_1 = \exp\{-2\beta J_1\}, \quad \sigma_x^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.10}$$

We note that in our case h_x can be a constant with values either h_1^* or h_3^* (see Section 4.1) and a function which is defined by (4.4).

Denote $\mathcal{M}_\mu = \pi_{\omega_\mu}(A)''$, where π_{ω_μ} is a GNS-representation associated with the state ω_μ.

According to the extremity of ω_μ we find that $\mathcal{M}_\mu$ is a factor.

Modular group σ_t^μ of $\mathcal{M}_\mu$ associated with ω_μ is similarly defined as (5.5) where we use the expression $\tilde{H}_\mu(V_n)$ instead of $\tilde{H}(V_n)$. Using (5.9), (5.10) and by similar argument as in the previous Section 5.1 we may prove the existence of the modular group σ_t^μ.

Lemma 5.4. Let the following condition be satisfied: there exist integers p and m_i, $i \in \{1,2,3\}$ and the smallest number $\delta \in (0,1)$ such that

$$\frac{p_1}{p_{11}} = \delta^{m_1}, \qquad \frac{p_2}{p_{22}} = \delta^{m_2}, \qquad \frac{p_1}{p_{22}} = \delta^{m_3}, \qquad \exp\{h_*\} = \delta^p, \tag{5.11}$$

then for the modular group $\sigma_t^{\omega_\mu}$ and the number $t_0 = -2\pi/\log\delta$, the equality holds

$$\sigma_{t_0}^{\omega_\mu} = Id.$$

Keeping in mind (5.9), (5.10) and repeating similar argument of the proof of Lemma 5.1 one can prove Lemma 5.4.

As in the proof of Theorem 5.3 using Lemma 5.4 instead of Lemma 5.1 one can prove the following

Theorem 5.5. Let the condition (5.11) be satisfied, then a von Neumann algebra $\mathscr{M}_\mu$ corresponding to an extreme $G_2^{(2)}$-periodic Gibbs measure μ of the model with ternary and binary interactions (2.1) on the Cayley tree Γ^2 is a factor of type III_δ. Otherwise $\mathscr{M}_\mu$ is a factor of type III_1.

6. DISCUSSION OF RESULTS

In the Ising model to each point x of the lattice there is assigned a spin variable $\sigma(x)$ taking its values $+1$ or -1. This model, at first considered as a ferromagnetic model, became a focus of active research and has various applications in many other fields of physics, chemistry, biology and even sociology. The model considered above (2.1) is a natural generalization of the Ising model. In refs. 26, 27 one gives "physical" motivations and actualities for exploring such models. In these and other papers the model (2.1) has been considered on such structures as Husimi tree[26, 27] and Kagome lattice.[26] We note, that in all in these works an exact solution of phase transitions was not obtained, one only gave solutions for the certain concrete values of parameters.

According to Theorem 3.1 a problem of describing limit Gibbs measures was reduced to the problem of description of the solutions of functional Eq. (3.4).

The proved Theorem 5.3 implies that at $J_1 = 0$ there is no phase transition for the considered model. We note that in ref. 16 the uniqueness of the solution of (3.4) has been proved only in the class of constant functions. It was known as a hypothesis that von Neumann algebras corresponding to physical systems with non-trivial interactions have only type III_1. The last Theorem 5.3 shows that mentioned hypothesis is not true for the considered model.

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